Study Guide 2

MATH 172 Lab: Sections 7 and 8

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Student's ID:-------------------------------- – Solution –

Note: *This study guide contains my practice questions that I think will be useful for preparing you for the second exam in Calculus II.*

Question 1: Evaluate the integral: $\int \cos(\sqrt[3]{x})\, dx$.

Let $w = \sqrt[3]{x}$

$w^3 = x$

Take the derivative of both sides $\Rightarrow 3w^2\, dw = dx$

$\int \cos(w)[3w^2\, dw]$

$\Rightarrow 3\boxed{\int w^2 \cos(w)\, dw}$

let's use table method as follows:

derivative Part	Integration
w^2	$\cos(w)$
$2w$	$-\sin(w)$ $\oplus$
2	$-\cos(w)$ $\ominus$
0	$\sin(w)$ $\oplus$

$\Rightarrow 3\int w^2 \cos(w)\, dw = -3w^2 \sin(w) + 2w\cos(w) + 6\sin(w) + C$

Thus, $\boxed{\int \cos(\sqrt[3]{x})\, dx = -3(\sqrt[3]{x})^2 \sin(\sqrt[3]{x}) + 2(\sqrt[3]{x})\cos(\sqrt[3]{x}) + 6\sin(\sqrt[3]{x}) + C}$

Question 2: Evaluate the integral: $\int \sqrt{\sin(x)}\cos^5(x)\, dx$.

$\Rightarrow \int \sqrt{\sin(x)}\cos^5 x\, dx = \int \sqrt{\sin(x)}\cos^4 x \cos x\, dx =$

$= \int \sqrt{\sin(x)}\,(1-\sin^2 x)^2 \cos x\, dx$ $\quad \boxed{\begin{array}{l} u = \sin(x) \\ du = \cos(x)\,dx \end{array}}$

$= \int u^{1/2}(1-u^2)^2\, du = \int u^{1/2}(1-2u^2+u^4)\, du =$

$= \int [u^{1/2} - 2u^{5/2} + u^{9/2}]\, du = \frac{u^{3/2}}{3/2} - 2\frac{u^{7/2}}{7/2} + \frac{u^{11/2}}{11/2} + C$

$= \frac{(\sin x)^{3/2}}{3/2} - 2\frac{(\sin x)^{7/2}}{7/2} + \frac{(\sin x)^{11/2}}{11/2} + C$ ☒

Question 3: Evaluate the integral: $\int \sqrt[3]{\tan(x)}\sec^6(x)\, dx$.

$\Rightarrow \int \sqrt[3]{\tan(x)}\sec^6(x)\, dx = \int \sqrt[3]{\tan(x)}\sec^4 x \sec^2 x\, dx$

$= \int \sqrt[3]{\tan(x)}\,(\tan^2 x + 1)^2 \sec^2 x\, dx$ $\quad \boxed{\begin{array}{l} u = \tan x \\ du = \sec^2 x\, dx \end{array}}$

$= \int u^{1/3}(u^2+1)^2\, du$

$= \int u^{1/3}(u^4 + 2u^2 + 1)\, du = \int [u^{13/3} + 2u^{7/3} + u^{1/3}]\, du$

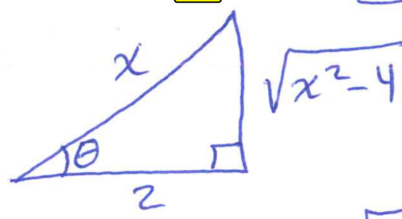
$= \frac{u^{16/3}}{16/3} + \frac{2u^{10/3}}{10/3} + \frac{u^{4/3}}{4/3} + C = \frac{(\tan x)^{16/3}}{16/3} + 2\frac{(\tan x)^{10/3}}{10/3} + \frac{(\tan x)^{4/3}}{4/3} + C$ ☒

$$\boxed{\begin{aligned}1+\tan^2\theta&=\sec^2\theta\\ \sec^2\theta-1&=\tan^2\theta\end{aligned}}$$

Question 4: Evaluate the integral using trigonometric substitution: $\int \frac{1}{x^3\sqrt{x^2-4}}\,dx$.

$x = 2\sec\theta \Longrightarrow \boxed{\sec\theta = \frac{x}{2}}$

$dx = 2\sec\theta\tan\theta\,d\theta$



$$\sqrt{x^2-4} = \sqrt{(2\sec\theta)^2-4} = \sqrt{4\sec^2\theta-4} = \sqrt{4(\sec^2\theta-1)} = 2\sqrt{\sec^2\theta-1} =$$

$$= 2\sqrt{\tan^2\theta} = \boxed{2\tan\theta}$$

$$\int \frac{1}{(2\sec\theta)^{\cancel{3}\,2}\cancel{(2\tan\theta)}}\cdot\cancel{(2\sec\theta\tan\theta)}\,d\theta = \frac{1}{8}\int\frac{1}{\sec^2\theta}\,d\theta = \frac{1}{8}\int\cos^2\theta\,d\theta$$

$$\boxed{\begin{aligned}\cos^2\theta &= \frac{1+\cos(2\theta)}{2}\\ \sin^2\theta &= \frac{1-\cos(2\theta)}{2}\end{aligned}}$$

$$= \frac{1}{8}\int\left[\frac{1}{2}+\frac{\cos(2\theta)}{2}\right]d\theta = \frac{1}{8}\left[\frac{1}{2}\theta+\frac{\sin(2\theta)}{4}\right]+C$$

$$= \frac{1}{16}\theta+\frac{2}{32}\sin(\theta)\cos(\theta)+C$$

We know
$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$$\boxed{\begin{aligned}&=\left[\frac{1}{16}\sec^{-1}\left(\frac{x}{2}\right)+\frac{1}{16}\,\frac{\sqrt{x^2-4}}{x}\cdot\frac{2}{x}\right]+C\\ &=\frac{1}{16}\sec^{-1}\left(\frac{x}{2}\right)+\frac{1}{8}\,\frac{\sqrt{x^2-4}}{x^2}+C\end{aligned}}$$

Question 5: Evaluate the integral: $\int \frac{4x}{(x^2-1)(x^2+1)}\,dx$.

$$\frac{4x}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$

$$4x = A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x-1)(x+1)$$

$x=1$: $4 = 4A \Rightarrow \boxed{A=1}$

$x=-1$: $-4 = -4B \Rightarrow \boxed{B=1}$

$x=0$: $0 = A-B-D$

$0 = 1-1-D \Rightarrow D = 1-1 = 0 \Rightarrow \boxed{D=0}$

$x=2$: $8 = 15A+5B+(2C+D)(3) \Rightarrow 8 = 15+5+6C \Rightarrow \boxed{C=-2}$

Thus, $\int\left[\frac{1}{x-1}+\frac{1}{x+1}-\frac{2x}{x^2+1}\right]dx = \boxed{\ln|x-1|+\ln|x+1|-\ln|x^2+1|+C}$

Question 6: Solve the following differential equation:

$$\frac{dy}{dx} = \boxed{\frac{(\overset{w}{2x+y})^2 - 1}{(\underset{w}{2x+y}) + 4} - 2}$$

(There is no need to write your solution as $y(x)$ in this problem)

Let $w = (2x+y)$

$\Rightarrow \frac{dw}{dx} = 2 + \frac{dy}{dx}$. Solve for $\frac{dy}{dx}$ as follows:

$\frac{dy}{dx} = \frac{dw}{dx} - 2 \Rightarrow$ Now, do the substitution:

$\frac{dw}{dx} \cancel{-2} = \frac{w^2-1}{w+4} \cancel{-2} \Rightarrow \frac{dw}{dx} = \frac{w^2-1}{w+4} \Rightarrow \frac{dw}{dx} \times \frac{w+4}{w^2-1}$

$\Rightarrow \frac{w+4}{w^2-1} dw = dx \Rightarrow \frac{w+4}{w^2-1} dw - dx = 0 \Rightarrow \boxed{\int \frac{w+4}{w^2-1} dw} - \int dx = \int 0$

use partial fractions

$\Rightarrow \int \frac{w+4}{w^2-1} dw - x = C \Rightarrow$ let's evaluate $\int \frac{w+4}{w^2-1} dw$ by partial fractions

Cover Method

$\Rightarrow \frac{w+4}{(w-1)(w+1)} = \frac{A}{w-1} + \frac{B}{w+1}$

$w=1$ $\quad$ $w=-1$

$A = \frac{1+4}{1+1} = \boxed{\frac{5}{2}}$

$B = \frac{-1+4}{-1-1} = \frac{3}{-2} = \boxed{-\frac{3}{2}}$

$\int\left(\left[\frac{5/2}{w-1}\right] + \left[\frac{-3/2}{w+1}\right]\right) dw =$

$\frac{5}{2}\ln|w-1| - \frac{3}{2}\ln|w+1| + C$

$= \frac{5}{2}\ln|w-1| - \frac{3}{2}\ln|(2x+y)+1| + C$

$= \frac{5}{2}\ln|(2x+y)-1| - \frac{3}{2}\ln|(2x+y)+1|$

$\Rightarrow \boxed{\frac{5}{2}\ln|(2x+y)-1| - \frac{3}{2}\ln|(2x+y)+1| - x = C}$

where C is a constant

Question 7: Evaluate the integral: $\int \frac{\ln(x+1)}{x^3} dx$.

$u = \ln(x+1)$ $\quad$ $dv = x^{-3} dx$

$du = \frac{1}{x+1} dx$ $\quad$ $v = \frac{x^{-2}}{-2}$

$\int \frac{\ln(x+1)}{x^3} dx = -\frac{1}{2}\ln(x+1)x^{-2} + \frac{1}{2}\int \frac{1}{x^2(x+1)} dx$

$= -\frac{1}{2x^2}\ln(x+1) + \frac{1}{2}\boxed{\int \frac{1}{x^2(x+1)} dx}$ partial fractions

$\Rightarrow \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$1 = Ax^2(x+1) + Bx(x+1) + C(x)(x^2)$

$x=0$: $\boxed{1 = B}$, $x=-1$: $\boxed{1 = C}$

$x=1$: $1 = 2A + 2B + C \Rightarrow \boxed{A = -1}$

$\Rightarrow \int\left[\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right] dx = -\ln|x| + \frac{x^{-1}}{-1} + \ln|x+1| + C$ ☑

Question 7: Continue:

Thus, $\int \frac{\ln(x+1)}{x^3} dx =$

$= \boxed{\frac{-1}{2x^2}\ln(x+1) - \frac{1}{2}\ln|x| - \frac{1}{2x} + \frac{1}{2}\ln|x+1| + C}$

this is the final answer. ☑

Question 8: Use the definition of improper integrals to evaluate the following integral: $\int_0^1 x\ln(x)dx$.

$$\int_0^1 x\ln(x)dx = \lim_{t\to 0^+} \boxed{\int_t^1 x\ln(x)dx}$$

$u = \ln x \qquad dv = x\,dx$

$du = \frac{1}{x} \qquad v = \frac{x^2}{2}$

$$\int x\ln(x)dx = \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int x\,dx$$

$$= \frac{1}{2}x^2\ln(x) - \frac{1}{2}\frac{x^2}{2} + C$$

$$\text{So, } \int_t^1 x\ln(x)dx = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2\Big|_t^1$$

$$= \left[\frac{1}{2}(1)\cancel{\ln(1)}^{0} - \frac{1}{4}(1)^2\right] - \left[\frac{1}{2}(t)\ln(t) - \frac{1}{4}t^2\right]$$

$$\lim_{t\to 0^+}\left[\cancel{\frac{-1}{2}t\ln(t)}^{0} + \cancel{\frac{1}{4}t^2}^{0} - \frac{1}{4}\right]$$

$$= \lim_{t\to 0^+}\left(-\frac{1}{4}\right) = \frac{-1}{4}$$

$\boxed{\text{Convergent}}$ by the definition of improper integrals. ☑

Question 9: Use the definition of improper integrals to evaluate the following integral: $\int_0^1 \frac{1}{\sqrt{1-x}}dx$.

$$\int_0^1 \frac{1}{\sqrt{1-x}}dx = \lim_{t\to 1^-} \boxed{\int_0^t \frac{1}{\sqrt{1-x}}dx}$$

$$\int \frac{1}{\sqrt{1-x}}dx \qquad u = 1-x,\ du = -dx$$

$$\Rightarrow -\int \frac{1}{\sqrt{u}}du = -\int u^{-1/2}du = \frac{-u^{1/2}}{1/2} = -2u^{1/2} = -2\sqrt{u} = -2\sqrt{1-x}$$

$$\text{So, } \int_0^t \frac{1}{\sqrt{1-x}}dx = -2\sqrt{1-x}\Big|_0^t = \boxed{-2\sqrt{1-t} + 2}$$

$$\lim_{t\to 1^-}\left[-2\sqrt{1-t} + 2\right] = -2\cancel{\sqrt{1-1}}^{0} + 2 = 2$$

$\boxed{\text{Convergent}}$ by the definition of improper integrals. ☑

Question 10: Evaluate the integral: $\int \frac{x^3}{x^2+1}\,dx$.
(**Hint:** Use long division)

Long Division

degree(numerator) ≥ degree(denominator)

$$\begin{array}{r} x \\ x^2+1 \overline{) \; x^3 \qquad} \\ \underline{-x^3 - x} \\ -x \end{array}$$

$$\int \frac{x^3}{x^2+1}\,dx = \int \left[x - \frac{x}{x^2+1}\right] dx =$$

$$= \frac{x^2}{2} - \frac{1}{2}\ln|x^2+1| + C$$

Thus, $\int \frac{x^3}{x^2+1}\,dx = \boxed{\frac{1}{2}x^2 - \frac{1}{2}\ln|x^2+1| + C}$ ☒

Question 11: Evaluate the integral: $\int \sec^3(x)\,dx$.

By parts

$$\boxed{\int \sec^3 x\,dx = \int \sec^2 x \sec(x)\,dx}$$

$u = \sec(x)$ $\qquad dv = \sec^2 x\,dx$

$du = \sec(x)\tan(x)\,dx$ $\qquad v = \tan(x)$

$$\int \sec^3(x)\,dx = \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx$$

We know

$$\boxed{\begin{array}{l} 1+\tan^2(x) = \sec^2(x) \\ \tan^2(x) = \sec^2(x) - 1 \end{array}}$$

$$\Downarrow$$

$$\int \sec(x)\left[\sec^2(x) - 1\right] dx$$

$$\Downarrow$$

$$\int \left[\sec^3(x) - \sec(x)\right] dx$$

$$\Downarrow$$

$$\boxed{\int \sec^3(x)\,dx - \int \sec(x)\,dx}$$

$$\int \sec^3(x)\,dx = \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx \Rightarrow$$

$2\int \sec^3(x)\,dx = \sec(x)\tan(x) + \int \sec(x)\,dx$

$= \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| + C$

Continue:

Thus, $\int \sec^3(x)\,dx = \frac{1}{2}(\sec(x)\tan(x)) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + C$

This is the final answer.

Question 12: Find the expression for the n^{th} term for each of the following sequences, and then determine whether it is convergent or divergent:

Part a: $\left\{2, \frac{3}{\sqrt{2}}, \frac{4}{\sqrt{3}}, \ldots\right\}$

$a_n = \frac{n+1}{\sqrt{n}} \Rightarrow \left\{\frac{n+1}{\sqrt{n}}\right\}_{n=1}^{\infty}$

$\lim_{n\to\infty} \frac{n+1}{\sqrt{n}} = \lim_{n\to\infty} \frac{n}{n^{1/2}} = \lim_{n\to\infty} n^{1/2} = \lim_{n\to\infty} \sqrt{n} = \sqrt{\infty} = \infty$

$\boxed{\text{Divergent}}$

Part b: $\left\{1, \frac{1}{8}, \frac{1}{27}, \frac{1}{64}, \ldots\right\}$

$a_n = \frac{1}{n^3} \Rightarrow \left\{\frac{1}{n^3}\right\}_{n=1}^{\infty}$

$\lim_{n\to\infty} \frac{1}{n^3} = \frac{1}{\infty} = 0$ $\boxed{\text{Convergent}}$

Question 13: Determine whether the following sequences are convergent or not:

Part a: $\left\{\frac{ln(n)}{n^2}\right\}_1^{\infty}$

(**Hint:** Use L'Hôpital's rule)

Since it's $\frac{\infty}{\infty}$, we use L'Hôpital's rule

$\lim_{n\to\infty} \frac{\ln(n)}{n^2} = \frac{\infty}{\infty} \Rightarrow \lim_{n\to\infty} \frac{\frac{1}{n}}{2n} = \frac{1}{2n^2} = \frac{1}{\infty} = 0$

$\boxed{\text{Convergent}}$

Part b: $\left\{e^{\sin\left(\frac{1}{n}\right)}\right\}_1^{\infty}$

$\lim_{n\to\infty} e^{\sin\left(\frac{1}{n}\right)} = e^{\sin\left(\frac{1}{\infty}\right)} = e^{\sin(0)} = e^0 = 1$ $\boxed{\text{Convergent}}$

Part c: $\left\{\frac{\sin(n)}{n^2}\right\}_1^{\infty}$

(**Hint:** Use sandwich theorem)

$\lim_{n\to\infty} \frac{\sin(n)}{n^2}$ here we need to use sequeeze (Sandwich) theorem as follows:

$-1 \le \sin(x) \le 1$

$-\frac{1}{n^2} \le \frac{\sin(x)}{n^2} \le \frac{1}{n^2}$

$\underbrace{\lim_{n\to\infty}\left(\frac{-1}{n^2}\right)}_{0} \le \lim_{n\to\infty} \frac{\sin(x)}{n^2} \le \underbrace{\lim_{n\to\infty}\frac{1}{n^2}}_{0}$

So, $\boxed{\lim_{n\to\infty} \frac{\sin(x)}{n^2} = 0}$

Thus, $\lim_{n\to\infty} \frac{\sin(n)}{n^2} = 0$

$\boxed{\text{Convergent}}$.

Question 14: Show the geometric series theorem.

$S_n = a + \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + \cancel{ar^{n-1}}$

$-$

$rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + ar^n$

let's subtract the 1st equation from the 2nd one, we obtain:

$\Rightarrow S_n - rS_n = a - ar^n \Longrightarrow S_n(1-r) = a(1-r^n)$

let's now divide both sides by $(1-r)$, we obtain:

$\frac{S_n\cancel{(1-r)}}{\cancel{(1-r)}} = a\frac{(1-r^n)}{(1-r)} \Longrightarrow \boxed{S_n = a\frac{(1-r^n)}{(1-r)}}$

So, $\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \Rightarrow |r|<1 \leftarrow \text{(Converge)} \\ \pm\infty & \text{if } |r|>1 \leftarrow \text{(diverge)} \end{cases}$

Good Luck in Exam 2

Best of Luck

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theorem: the geometric series: $a + ar + ar^2 + \cdots + ar^{n-1}$ is $\boxed{\text{convergent}}$ if $|r|<1$ and its sum is $S_n = a\frac{(1-\overset{0}{r^n})}{(1-r)} = \boxed{\frac{a}{1-r}}$. Otherwise, if $|r|>1$, then the geometric series is $\boxed{\text{divergent}}$. ☑