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Numerical Integration Lab

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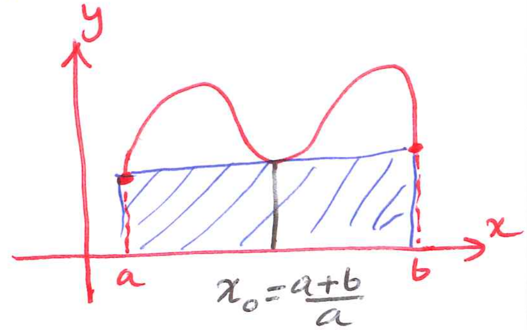
* What is the numerical integration?

Many techniques are taught in Calculus courses for the exact relation of integrals. but these techniques can seldom be used to evaluate integrals that occur in real-life problems. Exact techniques fail to solve many problems that arise in physical world; for these we need approximation methods (Numerical Integration).

* Basic Numerical Analysis Methods of Approximation:

① **Midpoint Rule**: If $f \in C^2[a,b]$, then a number say, ε, exists with $(a < \varepsilon < b)$ such that:

$$\int_a^b f(x)\,dx = \underbrace{(b-a) f\left(\frac{a+b}{2}\right)}_{\text{Approximation Part}} + \underbrace{\frac{f''(\varepsilon)}{24}(b-a)^3}_{\text{Error Part}}$$



$\Longrightarrow$ ①

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$$P_n(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \cdots$$

$$\cdots + f[x_0,\ldots,x_n](x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

$$|E_n(x)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n)\right|$$

$\boxed{\text{Error term}}$

$$P_0(x) = f[x_0] = f(\overset{\text{midpoint}}{x_0}) = f\left(\frac{a+b}{2}\right), \quad f(x) \approx P_0(x)$$

$$\int_a^b f(x)\,dx \approx \int_a^b P_0(x)\,dx = \int_a^b f\left(\frac{a+b}{2}\right)dx = f\left(\frac{a+b}{2}\right)\int_a^b 1\,dx = f\left(\frac{a+b}{2}\right)(b-a)$$

$$f(x) \approx P_n(x) \Longrightarrow \boxed{\int_a^b f(x)\,dx \approx \int_a^b P_n(x)\,dx}$$ this in general.

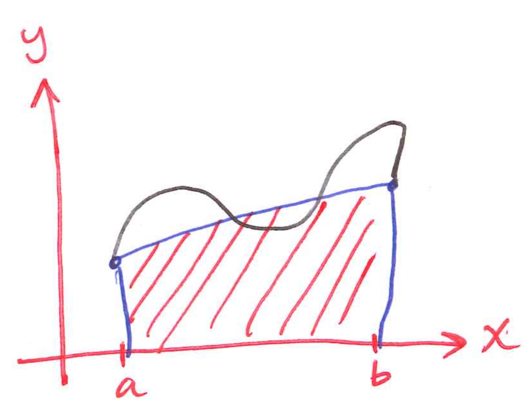
So, $$\int_a^b P_n(x)\,dx = \underbrace{\int f[x_0,x_1](x-x_0)}_{\text{Approximation}} + \underline{\int f[x_0,x_1,x_2](x-x_0)(x-x_1)}$$

$$\underbrace{+ \cdots}_{\uparrow\ \text{Error}}$$

② Trapezoidal Rule

If $f \in C^2[a,b]$, then a number, ξ, in (a,b) exists with

$$\int_a^b f(x)\,dx = \underbrace{(b-a)\frac{f(a)+f(b)}{2}}_{\text{Approximation}} - \underbrace{\frac{f''(\xi)}{12}(b-a)^3}_{\text{Error}}$$



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$P_1(x) = f[x_0] + f[x_0, x_1](x - x_0)$

$x_0 = a$

$x_1 = b$

$\int_a^b f(x)dx \approx \int_a^b P_1(x)dx$ this is in general.

③ Simpson's Rule

If $f \in C^2[a,b]$, then a number, ε, in (a,b) exists with

$$\int_a^b f(x)dx \approx \underbrace{\left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]}_{\text{Approximation}} + \underbrace{\frac{1}{90}\left(\frac{b-a}{2}\right)^5 \left| f^{(4)}(\varepsilon) \right|}_{\text{Error}}$$

In general, $\int_a^b P_n(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

Theorem: (Fundamental Theorem of Calculus)

Assume $f(x)$ is continuous on $[a,b]$, then

Part ①: $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

Part ②: $\int_a^b f(x)dx = F(b) - F(a)$ where F is an anti-derivative of f.