Mohammed Kaabar

Review of Convergence Tests

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* Increasing Sequences:

$\boxed{a_{n+1} \geq a_n}$ or

$\boxed{a_{n+1} - a_n \geq 0}$ or

$\Rightarrow \boxed{\frac{a_{n+1}}{a_n} \geq 1}$

Example 1: Show that the sequence $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ is increasing.

Solution:

Method I:

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1}$$

$$= \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)}$$

$$= \frac{\cancel{n^2} + \cancel{2n} + 1 - \cancel{n^2} - \cancel{2n}}{(n+2)(n+1)}$$

$$= \frac{1}{(n+2)(n+1)} > 0$$

So, it's increasing.

Method II: If the terms are ≥ 0, then:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{n+2}}{\frac{n}{n+1}} = \frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n} > 1$$

So, it's increasing.

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⇒ Method III: First derivative test

$$f(n)=\frac{n}{n+1} \Rightarrow f'(n)=\frac{(n+1)(1)-n(1)}{(n+1)^2}=\frac{\cancel{n}+1-\cancel{n}}{(n+1)^2}=\frac{1}{(n+1)^2}>0$$

$\Rightarrow f'(n)>0 \Rightarrow$ So, f is increasing. ■

* Definition: The sequence $\{a_n\}_{n=1}^{\infty}$ is $\boxed{\text{bounded}}$ if there exists a number $\boxed{M>0}$ such that $\boxed{|a_n|\leq M}$ for all n.

Example 2: $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ Is it a bounded sequence?!

Solution: $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} \Rightarrow \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ converges to 1

$\Rightarrow \boxed{\frac{1}{2}\leq a_n\leq 1} \Rightarrow \boxed{|a_n|\leq 1}$

* Theorem: Every bounded monotonic (increasing or decreasing) sequence $\boxed{\text{converges}}$.

* Convergence Tests:

I. Divergence Test

• Theorem: If the series $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{converges}}$, then $\lim_{n\to\infty} a_n=0$

OR: If $\lim_{n\to\infty} a_n\neq 0$, then the $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{diverges}}$

If p	then q
$p \longrightarrow q$	
If NOT $\sim q$	Then Not $\sim p$
$\sim q \longrightarrow \sim p$	

Example 3: Determine if it's convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{2n^2+n-3}{n^2+5}.$$

Solution: $\lim_{n\to\infty} \frac{2n^2+n-3}{n^2+5} = \frac{2\cancel{n^2}}{\cancel{n^2}} = 2 \neq 0$

So, it's $\boxed{\text{divergent}}$. ∎

Example 4: Determine if it's convergent or divergent:

$$\sum_{n=1}^{\infty} e^{1/n}.$$

Solution: $\lim_{n\to\infty} e^{1/n} = e^{1/\infty} = e^0 = 1 \neq 0$

So, it's $\boxed{\text{divergent}}$. ∎

II. Integral Test

Suppose f is $\boxed{\text{continuous}}$, $\boxed{\text{positive}}$, $\boxed{\text{decreasing}}$ function on $\boxed{[1,\infty)}$ and let $\boxed{a_n = f(n)}$. Then, the series $\boxed{\sum_{n=1}^{\infty} a_n}$ is convergent if and only if $\boxed{\int_1^{\infty} f(x)\,dx}$

Example 5: Determine if it's convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}.$$

Solution: $f(x) = \frac{1}{x^2+1}$ $\overbrace{\frac{1}{2}, \frac{1}{5}, \frac{1}{10}}^{\text{decreasing}}, \ldots$ So, f is positive,

$\Longrightarrow$

$\Rightarrow$ decreasing, and continuous everywhere $[1,\infty)$

So, $f' = \dfrac{0-(2x)}{(x^2+1)^2} = \dfrac{-2x}{(x^2+1)^2} < 0 \Rightarrow$ $\boxed{f \text{ is decreasing.}}$

Thus, $\displaystyle\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+1}\,dx$ $\rightarrow \tan^{-1}x$

$= \lim_{t\to\infty}\left[\underbrace{\tan^{-1}(t)}_{\pi/2} - \underbrace{\tan^{-1}(1)}_{\pi/4}\right]$

$= \boxed{\dfrac{\pi}{4}}$ convergent. ☒

* **Theorem**: The p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ $\boxed{\text{converges}}$ if $p>1$ and diverges if $p \le 1$.

Example 6: Determine if it's convergent or divergent:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Solution: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ Suppose $p=2$ which is $\boxed{p=2>1}$. So, by p-series definition, it's $\boxed{\text{convergent}}$. ☒

$\Rightarrow$

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III. Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are infinite series with positive terms.

1. If $\boxed{a_n \leq b_n}$ for all n and $\boxed{\sum b_n \text{ converges}}$, then $\boxed{\sum a_n \text{ converges}}$.
2. If $\boxed{a_n \geq b_n}$ for all n and $\boxed{\sum b_n \text{ diverges}}$, then $\boxed{\sum a_n \text{ diverges}}$.

IV. Limit Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are $\boxed{\text{infinite series}}$ with positive terms. If $\boxed{\lim_{n\to\infty} \frac{a_n}{b_n} = C > 0}$ where C is a finite number, then both series's $\boxed{\text{converge}}$ or both of them $\boxed{\text{diverge}}$.

Example 7: Determine if it's convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{\sin(n)+1}{n^2}$$

Solution: $-1 \leq \sin(n) \leq 1$ (Sandwich Theorem)

add 1 $\quad -1+1 \leq \sin(n)+1 \leq 1+1$

$$0 \leq \sin(n)+1 \leq 2$$

$\Longrightarrow$

$\Rightarrow$ Divide both sides by n^2, we obtain:

$$\Rightarrow \frac{0}{0} \le \frac{\sin(n)+1}{n^2} \le \frac{2}{n^2}$$

$$\Rightarrow 0 \le \frac{\sin(n)+1}{n^2} \le \frac{2}{n^2}$$

$\frac{2}{n^2} \to$ p-series, $p = 2 > 1$, convergent

So, $\sum_{n=1}^{\infty} \frac{\sin(n)+1}{n^2}$ is convergent. ☒

V. Ratio Test

Assume $\sum a_n$ is a series with positive terms and $\boxed{\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L}$, then if:

1. $\boxed{L > 1} \Rightarrow$ series $\boxed{\text{diverges}}$.
2. $\boxed{L < 1} \Rightarrow$ series $\boxed{\text{converges}}$.
3. $\boxed{L = 1} \Rightarrow$ $\boxed{\text{no conclusion}}$.

VI. Root Test

Assume $\sum a_n$ is a series with positive terms and $\boxed{\lim_{n\to\infty} \sqrt[n]{a_n} = L}$, then if: $\Longrightarrow$

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① $\boxed{L > 1} \Rightarrow$ series $\boxed{\text{diverges}}$.

② $\boxed{L < 1} \Rightarrow$ series $\boxed{\text{converges}}$.

③ $\boxed{L = 1} \Rightarrow$ $\boxed{\text{no conclusion}}$.

Example 8: Determine if it's convergent or not:

$$\sum_{n=1}^{\infty} \frac{n!}{e^n}$$

$\boxed{\text{Note: } (n+1)! = (n+1)n!}$

Solution: $\lim_{n\to\infty} \frac{\cancel{(n+1)!}^{\,n+1}}{e^{n+1}} \cdot \frac{\cancel{e^n}}{\cancel{n!}} = \lim_{n\to\infty} \left(\frac{n+1}{e}\right) = \infty > 1$

$e^{n+1} \to \cancel{e^n} \cdot e^1$

divergent ☑

Example 9: Determine if it's convergent or not:

$$\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n.$$

Solution:

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2\cancel{n}}{3\cancel{n}} = \frac{2}{3} < 1 \quad \text{Convergent}$$

☑

Example 10: Determine if it's convergent or not: $\sum_{n=1}^{\infty} \left(\frac{e^n}{n^2}\right)^n$.

Solution: $\lim_{n\to\infty} \sqrt[n]{\left(\frac{e^n}{n^2}\right)^n} = \lim_{n\to\infty} \frac{e^n}{n^2} = \frac{\infty}{\infty}$

$\xrightarrow{L'H} \lim_{n\to\infty} \frac{e^n}{2n} = \lim_{n\to\infty} \frac{e^n}{2} = \frac{\infty}{2} = \infty > 1$ divergent

☑